A Limit Theorem for Monotone Matrix Functions

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ABSTRACT

Given a symmetric $m \times m$ matrix function $Q(t)$ which decreases on some interval $(0, \varepsilon]$, $\varepsilon > 0$ [i.e., $Q(t_1) - Q(t_2)$ is nonnegative definite for $t_1 \leqslant t_2$] and which admits a factorization of the form $Q(t) = U(t)X^{-1}(t)$, where $U(t) \to U$, $X(t) = X$ as $t \to 0+$ with $\operatorname{rank}(U^T, X^T) = m$. Then it is shown that

$$\lim_{t \to 0+} X^T Q(t) X = U^T X, \quad \text{and} \quad \lim_{t \to 0+} c^T Q(t) c = \infty \quad \text{for all } c \notin \operatorname{Im} X.$$

Moreover, any monotone matrix function can be factorized as above.

1. INTRODUCTION AND NOTATION

Throughout we shall use the following notation: By ker, Im, rank, def, ind we denote respectively the kernel, image, rank, defect (that is, the dimension of the kernel), and negative index (that is, the number of the negative eigenvalues) of a matrix; and by I and $\operatorname{diag}(\alpha_1, \dots)$ we denote respectively the identity matrix and a diagonal matrix with diagonal elements $\alpha_1, \dots$. Moreover we write $Q_1 < Q_2$ $[Q_1 \leqslant Q_2]$ if Q_1 and Q_2 are symmetric (and real) and if $Q_2 - Q_1$ is positive definite [nonnegative definite]. *Monotonicity*

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of symmetric matrix functions $Q(t)$ is defined accordingly; e.g., $Q(t)$ *increases* if $Q(t_1) \leqslant Q(t_2)$ for $t_1 \leqslant t_2$. Of course, limits and differentiation of matrix functions $Q(t) = (q_{ij}(t))$ are always meant elementwise, e.g., $Q'(t) = (q'_{ij}(t))$.

In this paper we shall derive the following results on monotone matrix functions. Given any $m \times m$ matrix function $Q(t)$ on some one-sided interval $(0, \varepsilon]$, $\varepsilon > 0$, such that $Q(t)$ is (symmetric and) monotone on $(0, \varepsilon]$. Then $Q(t)$ can be factorized in the form $Q(t) = U(t)X^{-1}(t)$, where $U(t)$ and $X(t)$ satisfy the following properties on $(0, \delta] \subset (0, \varepsilon]$ for some $\delta > 0$: $U^T(t)X(t) \equiv X^T(t)U(t)$; $U(t) \to U$, $X(t) \to X$ as $t \to 0+$; $X^TU = U^TX$, $\operatorname{rank}(U^T, X^T) = m$; and $X(t)$ is, of course, invertible (regular) on $(0, \delta]$. This factorization is stated in Theorem 2 (Section 3). Moreover, if $Q(t) = U(t)X^{-1}(t)$ is decreasing on $(0, \delta]$, where $U(t)$ and $X(t)$ satisfy these properties, then

$$\lim_{t \to 0+} X^TQ(t)X = U^TX, \quad \text{and} \quad \lim_{t \to 0+} c^TQ(t)c = \infty \quad \text{for all } c \notin \operatorname{Im} X.$$

This limit result is the content of Theorem 1 (Section 2).

These results complement or even complete an earlier paper [3], which is also required for the proofs. Our result can be applied e.g. to handle matrix functions which occur in connection with eigenvalue problems for linear self-adjoint differential systems or Riccati matrix differential equations (see [1] and [4]), and which are also of interest in control theory (optimal linear regulator). The relation to the optimal linear regulator in control theory will be described in a forthcoming paper. Here we describe briefly the connection with linear self-adjoint differential systems and Riccati equations.

Suppose that $A(t)$, $B(t)$, $C(t)$, and $C_0(t)$ are $m \times m$ matrix functions, which are piecewise continuous on $\mathbb{R}$, such that $B(t)$, $C(t)$, and $C_0(t)$ are symmetric, and such that $B(t)$ and $C_0(t)$ are nonnegative definite on $\mathbb{R}$. Then, for a fixed parameter ("eigenvalue") $\lambda \in \mathbb{R}$, we consider the solution $Q(t) = Q(t; \lambda)$ of the *Riccati matrix differential equation*

$$\dot{Q} + A^TQ + QA + QBQ - C + \lambda C_0 = 0 \qquad (*)$$

with the (fixed with respect to λ) initial condition

$$Q(t_0; \lambda) = Q_0,$$

where Q_0 is any given symmetric $m \times m$ matrix, and where $t_0 \in \mathbb{R}$ is also given. Now, for fixed $t > t_0$, it follows (see [1, (2.4) and (2.9)]) that $Q(t; \lambda)$, as a matrix function in λ, is decreasing. Moreover, the factorization of Q needed above occurs naturally, namely,

$$Q(t) = Q(t;\lambda) = U(t)X^{-1}(t) = U(t;\lambda)X^{-1}(t;\lambda),$$

where the $m \times m$ matrix functions $X(t) = X(t;\lambda)$ and $U(t) = U(t;\lambda)$ are the solution of the *linear self-adjoint differential system*

$$\dot{X} = AX + BU, \qquad \dot{U} = (C - \lambda C_0)X - A^T U, \tag{$**$}$$

which satisfies the initial condition

$$X(t_0;\lambda) = I, \qquad U(t_0;\lambda) = Q_0.$$

Hence, our result here may be applied to derive the asymptotic behavior of $Q(t;\lambda)$ as $\lambda \to \lambda_0$ (where $t > t_0$ is fixed) when $X(t;\lambda_0)$ is noninvertible. Actually, this limit result is already contained in [4, Section 4], and the main contribution of this paper is to show that (essentially) only the monotonicity of $Q(t;\lambda)$, as a consequence of the Riccati equation $(*)$ [and of the assumption that $B(t)$ and $C_0(t)$ are nonnegative], is needed, and nothing else (not even differentiability).

Let us mention also that the dependence of $Q(t;\lambda)$ on λ plays an important role in deriving e.g. oscillation theorems on eigenvalue problems, which are built up from $(**)$ and $2n$ additional linearly independent and self-adjoint boundary conditions. This oscillation theory (see [1 and 4]) also needs the following result on the asymptotic behavior of solutions of $(**)$ as t varies (which is already contained in [3]): If, for fixed $\lambda \in \mathbb{R}$, $X_1(t)$, $U_1(t)$ and $X_2(t)$, $U_2(t)$ are so-called "normalized conjoined bases" of $(**)$ [which means that they solve $(**)$ with $X_i^T(t')U_i(t') = U_i^T(t')X_i(t')$ and $X_1^T(t')U_2(t') - U_1^T(t')X_2(t') = I$ for some $t' \in \mathbb{R}$], then $H(t) = X_1^{-1}(t)X_2(t)$ is increasing on $\mathbb{R}$ (see [1, (2.5)]), and, as above, our result here describes the asymptotic behavior of $H(t)$ as $t \to t_0$ when $X_1(t_0)$ is noninvertible.

2. THE LIMIT THEOREM

Our main result in this section is

THEOREM 1. *Let there be given $m \times m$ matrices $U(t)$ and $X(t)$ for*

$t \in (0, \varepsilon]$, $\varepsilon > 0$, *such that the following holds*:

$$U^T(t)X(t) = X^T(t)U(t) \qquad \textit{for} \quad t \in (0, \varepsilon], \tag{1a}$$

$$U(t) \to U, \quad X(t) \to X \qquad \textit{as} \quad t \to 0+, \tag{1b}$$

$$\operatorname{rank}(U^T, X^T) = m \quad \textit{and} \quad U^T X = X^T U, \tag{2}$$

$$X(t) \textit{ is regular for } t \in (0, \varepsilon]; \tag{3}$$

and

$$Q(t) = U(t) X^{-1}(t) \textit{ is monotone on } (0, \varepsilon].$$

Then we have that

$$\lim_{t \to 0+} X^T Q(t) X = X^T U; \tag{5}$$

and

$$\lim_{t \to 0+} c^T Q(t) c = \infty \, [-\infty] \qquad \textit{for all} \quad c \notin \operatorname{Im} X \tag{6}$$

if $Q(t)$ *decreases* [*increases*] *on* $(0, \varepsilon]$.

REMARK 1. Observe that (1) implies that $Q(t)$ is symmetric on $(0, \varepsilon]$ and also that $U^T X = X^T U$ [which is part of (2)]. Of course, the corresponding result holds for left hand limits.

Our proof will use results from [3, 4], but in particular the following lemma.

LEMMA 1. *Given* $m \times m$ *matrices* U, X, Q_1, *and* Q_2 *such that* X, U *satisfy* (2), *let* $\tilde{U} = XK^{-1}$, $\tilde{X} = -UK^{-1}$, *where* $K = U^T U + X^T X$. *Moreover, assume that* Q_1 *and* Q_2 *are symmetric with* $Q_1 \leqslant Q_2$, *that* $\tilde{U} - Q_1 \tilde{X}$ *and* $\tilde{U} - Q_2 \tilde{X}$ *are regular, and that*

$$\operatorname{ind} \tilde{X}^T \left(\tilde{U} - Q_1 \tilde{X}\right) = \operatorname{ind} \tilde{X}^T \left(\tilde{U} - Q_2 \tilde{X}\right). \tag{7}$$

Then $\tilde{U} - \{Q_1 + t(Q_2 - Q_1)\}\tilde{X}$ *is regular for* $t \in [0, 1]$, *and* $S_2 \leqslant S_1$, *where* $S_j = S(Q_j) = (\tilde{U} - Q_j \tilde{X})^{-1}(U_j - Q_j X)$ *for* $j = 1, 2$.

Proof. Observe first that K is regular by (2), so that $\tilde{U}$ and $\tilde{X}$ are well defined. The definition of $\tilde{U}$, $\tilde{X}$, and K implies that

$$\begin{pmatrix} \tilde{U}^T & -\tilde{X}^T \\ -U^T & X^T \end{pmatrix}\begin{pmatrix} X & \tilde{X} \\ U & \tilde{U} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} X & \tilde{X} \\ U & \tilde{U} \end{pmatrix}\begin{pmatrix} \tilde{U}^T & -\tilde{X}^T \\ -U^T & X^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

and we obtain the identities (see also the notion of normalized conjoined bases in [4, Definition 2 and Equation (7)]):

$$U^TX = X^TU, \qquad \tilde{U}^T\tilde{X} = \tilde{X}^T\tilde{U}, \qquad X\tilde{X}^T = \tilde{X}X^T, \qquad U\tilde{U}^T = \tilde{U}U^T, \quad \text{(8a)}$$

$$\tilde{U}^TX - \tilde{X}^TU = X\tilde{U}^T - \tilde{X}U^T = I. \qquad \text{(8b)}$$

Next, we consider the matrix

$$M(t) = \tilde{X}^TR(t), \qquad \text{where} \quad R(t) = \tilde{U} - \{Q_1 + t(Q_2 - Q_1)\}\tilde{X}$$

for $t \in [0, 1]$. Then, by (2), (8), and our assumptions, $M(t)$ is symmetric and $\operatorname{rank}(\tilde{X}^T, R^T(t)) = m$, and [1, Proposition A1] or [4, Proposition A1] yields that

$$\ker M(t) = \ker \tilde{X} \oplus \ker R(t) \qquad \text{for} \quad t \in [0,1]. \qquad (9)$$

This and the regularity of the matrices $R(0) = \tilde{U} - Q_1\tilde{X}$ and $R(1) = \tilde{U} - Q_2\tilde{X}$ imply that

$$\ker M(0) = \ker M(1) = \ker \tilde{X}. \qquad (10)$$

By assumption $Q_2 \geqslant Q_1$, so that

$$M'(t) = -\tilde{X}^T(Q_2 - Q_1)\tilde{X} \leqslant 0.$$

Hence, $M(t)$ is decreasing, so that (by [4, Proposition A3]) its eigenvalues (suitably enumerated) are continuous and decreasing on $[0, 1]$. Now, by (10) and the assumption (7), $M(0)$ and $M(1)$ have the same numbers of negative and positive eigenvalues, so that none of the eigenvalues of $M(t)$ changes its sign on [0, 1] by the monotonicity. Hence, $\ker M(t) = \ker \tilde{X}$ for $t \in [0, 1]$, so that, by (9), $R(t)$ is regular, and

$$\tilde{S}(t) = R^{-1}(t)\left[U - \{Q_1 + t(Q_2 - Q_1)\}X\right]$$

exists for $t \in [0, 1]$ with $\tilde{S}(0) = S_1$, $\tilde{S}(1) = S_2$. Finally, $\tilde{S}(t)$ is symmetric by (8), and a simple calculation shows that

$$\tilde{S}'(t) = -R^{-1}(t)(Q_2 - Q_1)\{R^{-1}(t)\}^T \leqslant 0$$

for $t \in [0, 1]$. Thus, $S_1 = \tilde{S}(0) \geqslant \tilde{S}(1) = S_2$, which completes the proof. ■

Proof of Theorem 1. Let $\tilde{U}$, $\tilde{X}$, and K be defined as in Lemma 1, so that (8) holds. Moreover, let

$$S(t) = \{U^T X(t) - X^T U(t)\}\{\tilde{U}^T X(t) - \tilde{X}^T U(t)\}^{-1}, \tag{11}$$

where, by (1), (2), and (8),

$$U^T X(t) - X^T U(t) \to 0, \quad \tilde{U}^T X(t) - \tilde{X}^T U(t) \to I \qquad \text{as} \quad t \to 0+,$$

so that $S(t)$ is well defined on $(0, \delta] \subset (0, \varepsilon]$ with $\delta > 0$ sufficiently small. Now, (3) and the definition of $Q(t)$ [by (4)] imply that $S(t)$ is symmetric and that

$$S(t) = \{\tilde{U} - Q(t)\tilde{X}\}^{-1}\{U - Q(t)X\} \qquad \text{for} \quad t \in (0, \delta]. \tag{12}$$

It follows from (8) and (12) that

$$\begin{aligned} U - \tilde{U}S(t) &= \left\{U\left[\tilde{U}^T - \tilde{X}^T Q(t)\right] - \tilde{U}\left[U^T - X^T Q(t)\right]\right\}\left\{\tilde{U}^T - \tilde{X}^T Q(t)\right\}^{-1} \\ &= Q(t)\left\{\tilde{U}^T - \tilde{X}^T Q(t)\right\}^{-1} \end{aligned}$$

and that

$$X - \tilde{X}S(t) = \left\{X\left[\tilde{U}^T - \tilde{X}^T Q(t)\right] - \tilde{X}\left[U^T - X^T Q(t)\right]\right\}\left\{\tilde{U}^T - \tilde{X}^T Q(t)\right\}^{-1}$$

$$= \left\{\tilde{U}^T - \tilde{X}^T Q(t)\right\}^{-1}.$$

Hence, $X - \tilde{X}S(t)$ is regular, and

$$Q(t) = \left\{U - \tilde{U}S(t)\right\}\left\{X - \tilde{X}S(t)\right\}^{-1} \qquad \text{for} \quad t \in (0, \delta]. \tag{13}$$

By (4), we may assume that $Q(t)$ decreases on $(0, \varepsilon]$. Then $\tilde{X}^T\{\tilde{U} - Q(t)\tilde{X}\}$ is increasing on $(0, \varepsilon]$, and we obtain for sufficiently small $\delta > 0$:

(i) $\operatorname{ind} \tilde{X}^T\{\tilde{U} - Q(t)\tilde{X}\}$ is constant on $(0, \delta] \subset (0, \varepsilon]$;
(ii) $S(t_1) \leqslant S(t_2)$ for $0 < t_1 \leqslant t_2 \leqslant \delta$, by Lemma 1 and (i), since $Q(t_1) \geqslant Q(t_2)$; and
(iii) $0 \leqslant S(t)$ for $0 < t \leqslant \delta$ by (ii), since $\lim_{t \to 0+} S(t) = 0$.

Because of (iii) [and (8)] we can apply [3, Theorem 1] (observe that $S(t) > 0$, which is assumed in [3], may be replaced by $S(t) \geqslant 0$ and the regularity of $X^T - S(t)\tilde{X}^T$), which yields that

$$\lim_{t \to 0+} X^T\left\{X^T - S(t)\tilde{X}^T\right\}^{-1} S(t) = 0.$$

Hence, by (13) and (8),

$$X^T U - X^T Q(t) X$$

$$= X^T\left\{X^T - S(t)\tilde{X}^T\right\}^{-1}\left\{\left[X^T - S(t)\tilde{X}^T\right]U - \left[U^T - S(t)\tilde{U}^T\right]X\right\}$$

$$= X^T\left\{X^T - S(t)\tilde{X}^T\right\}^{-1} S(t) \to 0 \qquad \text{as} \quad t \to 0+,$$

which yields our assertion (5). The assertion (6) will follow under weaker assumptions from Proposition 1 below. ■

REMARK 2. If, in addition, $Q(t)$ is differentiable, then Lemma 1 is not needed, since the monotonicity of $S(t)$ [see (ii) above] will follow immediately from (12) by simple differentiation. While the monotonicity of $Q(t)$ [or

$S(t)$] is essential for assertion (5) (see [3, Section 5] and the discussion in Section 4 below), this assumption may be weakened so that the assertion (6) still holds. This can be seen from the following proposition.

PROPOSITION 1. *Let there be given $m \times m$ matrices $U(t)$ and $X(t)$ for $t \in (0, \varepsilon]$, $\varepsilon > 0$, satisfying* (1), (2), (3). *Then, if $\mu_j(t)$ denote the eigenvalues of $Q(t)$ with $|\mu_1(t)| \leqslant \cdots \leqslant |\mu_m(t)|$, we have that*

$$\begin{aligned} &\mu_j(t) = O(1) \quad \textit{as} \quad t \to 0+ \quad \textit{for} \quad j = 1, \dots, r, \quad \textit{where } r = \operatorname{rank} X, \\ &|\mu_j(t)| \to \infty \quad \textit{as} \quad t \to 0+ \quad \textit{for} \quad j = r+1, \dots, m; \end{aligned} \tag{14}$$

and

$$\lim_{t \to 0+} c^T Q(t) c = \infty \qquad \textit{for all} \quad c \notin \operatorname{Im} X, \tag{6}$$

if we assume additionally

$$Q(t) \geqslant -\alpha I \quad \textit{on } (0, \varepsilon] \qquad \textit{for some} \quad \alpha > 0. \tag{4'}$$

Proof. For the proof of (14) we use the minimum-maximum principle (see e.g. [5] or [4, Proposition A5]) as follows. Let $W_1(t) = \{X(t)c \mid c \in \ker X\}$, so that $\dim W_1(t) = m - r = \operatorname{def} X$ by (3). Since $Ud \neq 0$ for all $d \in \ker X \setminus \{0\}$ (by [3, Proposition A1]), we can conclude that there exist $\delta > 0$, $\eta > 0$ such that

$$\inf_{d \in W_1(t)} \frac{|Q(t)d|}{|d|} = \inf_{x \in \ker X} \frac{|U(t)c|}{|(X(t) - X)c|} \geqslant \frac{\eta}{\gamma(t)} \qquad \text{for} \quad t \in (0, \delta],$$

where $\gamma(t) = \|X(t) - X\| \to 0$ as $t \to 0+$ by (1). Hence, $|\mu_j(t)| \to \infty$ for $j = r+1, \dots, m$. Next, let $W_2(t) = \{X(t)c \mid c \in \operatorname{Im} X^T\}$, so that $\dim W_2(t) = r = \operatorname{rank} X$. By (1) there exist $\delta > 0$, $\eta > 0$ such that

$$\sup_{d \in W_2(t)} \frac{|Q(t)d|}{|d|} = \sup_{c \in \operatorname{Im} X^T} \frac{|U(t)c|}{|X(t)c|} \leqslant \eta \qquad \text{for} \quad t \in (0, \delta].$$

Hence, $\mu_j(t) = O(1)$ for $j = 1, \dots, r$, which yields (14).

For the proof of (6), observe that (4′) and what we have already shown imply that, for $t \to 0+$,

$$\mu_j(t) = O(1) \text{ for } j=1,\dots,r, \text{ and } \mu_j(t) \to \infty \quad \text{for } j=r+1,\dots,m.$$

Let $T(t)$ be an orthogonal matrix with

$$T^T(t)Q(t)T(t) = D(t) = \mathrm{diag}(\mu_1(t),\dots,\mu_m(t)) = \begin{array}{c} \begin{pmatrix} D_1(t) & 0 \\ 0 & D_2(t) \end{pmatrix} \begin{array}{l} \}r \\ \}m-r \end{array} \end{array}$$

where $D_1(t) = O(1)$, $D_2(t) \to \infty$ as $t \to 0+$. Now, assume that (c_k) is any convergent sequence in $\mathbb{R}^m$ with limit c, and that (t_k) is any sequence in $\mathbb{R}$ with $t_k \to 0+$ and

$$c_k^T Q(t_k) c_k = O(1) \qquad \text{as} \quad k \to \infty.$$

By compactness we may assume that $T(t_k) \to T$ where T is orthogonal. Thus,

$$d_k = T^T(t_k)c_k \to d = T^T c \qquad \text{as} \quad k \to \infty,$$

and

$$O(1) = c_k^T Q(t_k) c_k = d_k^T \begin{pmatrix} D_1(t_k) & 0 \\ 0 & D_2(t_k) \end{pmatrix} d_k$$

implies that

$$c = T \begin{pmatrix} d_1 \\ 0 \end{pmatrix} \qquad \text{with} \quad d_1 \in \mathbb{R}^r.$$

Since $X(t_k)c \to Xc$ and $c^T X^T(t_k) Q(t_k) X(t_k) c \to c^T X^T U c = O(1)$ as $k \to \infty$ for any sequence $t_k \to 0+$ by (1), we obtain that

$$\operatorname{Im} X \subset \operatorname{Im} T \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{array}{l} \}r \\ \}m-r \end{array},$$

and we have equality because of equal ranks. Hence, for any $c \in \mathbb{R}^m$ we have that $c^T Q(t_k)c = O(1)$ for some sequence $t_k \to 0+$ implies

$$c \in \operatorname{Im} T \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \operatorname{Im} X,$$

which yields (6). ■

REMARK 3. Of course, the assertion (6) becomes false in general without the assumption (4′) that $Q(t)$ is bounded from below (see Section 4).

3. FACTORIZATION OF MONOTONE MATRIX FUNCTIONS

In this section we show that every monotone matrix function $Q(t)$ can be written in the form $Q(t) = U(t)X^{-1}(t)$, where the matrices $U(t)$ and $X(t)$ are as in Theorem 1.

THEOREM 2. *Assume that $Q(t)$ is monotone on $(0, \varepsilon]$, $\varepsilon > 0$. Then there exist matrices $U(t)$ and $X(t)$ such that* (1), (2), *and* (3) *hold, and such that $Q(t) = U(t)X^{-1}(t)$ on some interval $(0, \delta] \subset (0, \varepsilon]$ with $\delta > 0$.*

Proof. We may assume that $Q(t)$ decreases on $(0, \varepsilon]$. Now, let $\mu_j(t)$ denote the eigenvalues of $Q(t)$, and denote by $T(t) = (d_1(t), \dots, d_m(t))$ an orthogonal matrix with columns $d_j(t)$ such that, for $t \in (0, \varepsilon]$,

$$\mu_1(t) \leqslant \cdots \leqslant \mu_m(t), \quad T^T(t)Q(t)T(t) = D(t) = \operatorname{diag}(\mu_1(t), \dots, \mu_m(t)). \tag{15}$$

First, we prove two auxiliary lemmas, and then Theorem 2 is a consequence of the subsequent proposition, where $U(t)$ and $X(t)$ are constructed. ■

LEMMA 2. *The following assertions hold:*

(i) $\mu_j(t)$ is decreasing on $(0, \varepsilon]$ for $j = 1, \dots, m$;
(ii) $T(t_k) \to T$ as $k \to \infty$ for some sequence $0 < t_k \to 0$ with an orthogonal matrix $T = (d_1, \dots, d_m)$ (with columns d_j);
(iii) there exists $r \in \{0, 1, \dots, m\}$ such that

$$\mu_j(t) \to \mu_j < \infty \quad \textit{for} \quad j = 1, \dots, r,$$
$$\mu_j(t) \to \infty \quad \textit{for} \quad j = r + 1, m$$

as $t \to 0+$;

(iv) $T_0^T Q(t) T_0 \to D_0$ as $t \to 0+$, where $T_0 = (d_1, \ldots, d_r, 0, \ldots, 0)$ and where $D_0 = \operatorname{diag}(\mu_1, \ldots, \mu_r, 0, \ldots, 0)$; and
(v) $c^T Q(t) c \to \infty$ as $t \to 0+$ for all $c \notin \operatorname{Im} T_0$.

Proof. The fact that $Q(t)$ decreases, the definition of $\mu_j(t)$ and $T(t)$ [i.e. (15)], and compactness yield at once assertions (i) (see e.g. [4, Proposition A3]), (ii), and (iii).

For the proof of (iv) let

$$T_0(t) = (d_1(t), \ldots, d_r(t), 0, \ldots, 0)$$

so that $T_0(t_k) \to T_0$ as $k \to \infty$. Then, for $c = (c_j) \in \mathbb{R}^m$, we have that

$$\{T_0 - T_0(t)\}c = T(t)\gamma \qquad \text{with} \quad \gamma = \gamma(t) = (\gamma_j(t)),$$

where $|\gamma(t_k)| \leqslant \|T_0 - T_0(t_k)\|\,|c| \to 0$ as $k \to \infty$ [observe that $T(t)$ is orthogonal, so that $\|T(t)\| \equiv 1$]. Hence, by (iii),

$$\begin{aligned} c^T T_0^T Q(t_k) T_0 c &= \sum_{j=1}^{r} \mu_j(t_k)(c_j + \gamma_j)^2 + \sum_{j=r+1}^{m} \mu_j(t_k)\gamma_j^2 \\ &\geqslant c^T D_0 c + \sum_{j=1}^{r} \mu_j(t_k)\left\{(c_j + \gamma_j)^2 - c_j^2\right\} \to c^T D_0 c \end{aligned}$$

as $k \to \infty$, uniformly for $c \in \mathbb{R}^m$ with $|c| = 1$. This and the monotonicity of $Q(t)$ imply that

$$\varliminf_{t \to 0+} T_0^T Q(t) T_0 \geqslant D_0.$$

Since $Q(t)$ decreases, we have that, for all $t \in (0, \varepsilon]$,

$$T_0^T Q(t) T_0 = \lim_{k \to \infty} T_0^T(t_k) Q(t) T_0(t_k) \leqslant \varlimsup_{k \to \infty} T_0^T(t_k) Q(t_k) T_0(t_k) = D_0,$$

and therefore (iv) holds.

Finally, for the proof of (v) consider $c \notin \operatorname{Im} T_0$. Then $c = T(t)\gamma(t)$ with $\gamma(t_k) \to \gamma = (\gamma_j) = T^T c$, so that $\sum_{j=r+1}^m \gamma_j^2 > 0$ as $k \to \infty$. Hence, by (iii),

$$c^T Q(t_k) c \geqslant \sum_{j=1}^r \mu_j(t_k)\gamma_j^2(t_k) + \mu_{r+1}(t_k) \sum_{j=r+1}^m \gamma_j^2(t_k)$$

$$\to \infty \qquad \text{as} \quad k \to \infty,$$

and this yields (v) by the monotonicity of $Q(t)$ again. ■

LEMMA 3. *Let $r \in \{0, \dots, m\}$ and $\mu_1, \dots, \mu_r$ be as in* (iii) *of Lemma* 2, *and assume that (t_k) and (τ_k) are any sequences such that $0 < t_k \to 0$, $0 < \tau_k \to 0$, $T(t_k) \to T$, and $T(\tau_k) \to T'$. Then $T' = TE$ with an orthogonal matrix E of the form*

$$E = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \begin{matrix} \}r \\ \}m-r \end{matrix}$$

such that

$$D_1 = \operatorname{diag}(\mu_1, \dots, \mu_r) = E_1 D_1 E_1^T = E_1^T D_1 E_1. \tag{16}$$

Proof. Of course, $E = T^T T'$ is orthogonal, since T and T' are orthogonal, and Lemma 2 [(iv) and (v)] implies that

$$T_0' = (d_1', \dots, d_r', 0, \dots, 0) \in \operatorname{Im} T_0 \qquad \text{for} \quad T_0 = (d_1, \dots, d_r, 0, \dots, 0)$$

where d_j' $[d_j]$ denote the columns of T' $[T]$ as above. Let

$$\mathfrak{E} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \begin{matrix} \}r \\ \}m-r \end{matrix}.$$

Similarly, $T' = (T_{ij}')$, $T = (T_{ij})$ so that $T = T_0 + \tilde{T}_0$ with

$$T_0 = \begin{pmatrix} T_{11} & 0 \\ T_{12} & 0 \end{pmatrix}$$

and

$$\tilde{T}_0 = \begin{pmatrix} 0 & T_{21} \\ 0 & T_{22} \end{pmatrix} = (0, \dots, 0, d_{r+1}, \dots, d_m).$$

Then

$$\begin{pmatrix} T'_{11} \\ T'_{21} \end{pmatrix} = T \begin{pmatrix} E_{11} \\ E_{21} \end{pmatrix} = T_0 \begin{pmatrix} E_{11} \\ 0 \end{pmatrix} + \tilde{T}_0 \begin{pmatrix} 0 \\ E_{21} \end{pmatrix} \in \operatorname{Im} T_0$$

yields that

$$\tilde{T}_0 \begin{pmatrix} 0 \\ E_{21} \end{pmatrix} = 0,$$

i.e., $T_{21} E_{21} = 0$ and $T_{22} E_{21} = 0$. Hence, $E_{21} = 0$. Interchanging the roles of T and T', it follows in the same way that $E_{12} = 0$, so that

$$E = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}.$$

Now, Lemma 2(iv) implies that

$$\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} = D_0 = \lim_{t \to 0+} T_0^T Q(t) T_0 = \lim_{t \to 0+} T_0'^T Q(t) T_0' = E^T D_0 E.$$

Hence, $D_1 = E_1^T D_1 E_1 = E_1 D_1 E_1^T$, which is (16). ■

PROPOSITION 2. *Assume that $Q(t)$ is monotone on $(0, \varepsilon]$, and let $T(t), \mu_1(t), \dots, \mu_m(t)$ be as in* (15). *Moreover, let $r \in \{0, \dots, m\}$ and $\mu_1, \dots, \mu_r$ be as in* (iii) *of Lemma* 2, *and put*

$$D_1 = \operatorname{diag}(\mu_1, \dots, \mu_r),$$

$$D(t) = \begin{pmatrix} D_1(t) & 0 \\ 0 & D_2(t) \end{pmatrix} \begin{matrix} \}r \\ \}m-r \end{matrix},$$

$$T(t) = \Big(\underbrace{T_1(t)}_{r}, \underbrace{T_2(t)}_{m-r} \Big).$$

Suppose that

$$T(t_k) \to T = (\underbrace{T_1}_{r}, \underbrace{T_2}_{m-r})$$

for some sequence $0 < t_k \to 0$ *as* $k \to \infty$. *Then the following assertions hold for sufficiently small* $\delta > 0$:

(i) $T_1(t)T_1^T(t) \to T_1T_1^T$, $T_2(t)T_2^T(t) \to T_2T_2^T$, and $T_1(t)D_1(t)T_1^T(t) \to T_1D_1T_1^T$ as $t \to 0+$;

(ii) $U(t) \to U = T_1D_1T_1^T + T_2T_2^T$, $X(t) \to X = T_1T_1^T$ as $t \to 0+$, where $U(t) = T_1(t)D_1(t)T_1^T(t) + T_2(t)T_2^T(t)$, and where $D_2(t)$ and $X(t) = T_1(t)T_1^T(t) + T_2(t)D_2^{-1}(t)T_2^T(t)$ are regular on $(0, \delta]$;

(iii) $Q(t) = U(t)X^{-1}(t)$ for $t \in (0, \delta]$; and

(iv) $\operatorname{rank}(U^T, X^T) = m$, $U^TX = X^TU$ (i.e. (2)).

Proof. By (iii) of Lemma 2 we may choose $\delta > 0$ such that $D_2(t)$ is regular on $(0, \delta] \subset (0, \varepsilon]$. First, assume that (τ_k) is any sequence with

$$0 < \tau_k \to 0, \quad T(\tau_k) \to T' = (\underbrace{T_1'}_{r}, \underbrace{T_2'}_{m-r}) \quad \text{as} \quad k \to \infty$$

as in Lemma 3. Hence, by Lemma 3,

$$T' = T\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = (T_1E_1, T_2E_2)$$

such that

$$E_1E_1^T = I_{r\times r}, \qquad E_2E_2^T = I, \quad D_1 = E_1D_1E_1^T = E_1^TD_1E_1,$$

and therefore

$$T_1(\tau_k)T_1^T(\tau_k) \to T_1'T_1'^T = T_1T_1^T, \qquad T_2(\tau_k)T_2^T(\tau_k) \to T_2T_2^T,$$

and

$$T_1(\tau_k)D_1(\tau_k)T_1^T(\tau_k) \to T_1'D_1T_1'^T = T_1E_1D_1E_1^TT_1^T = T_1D_1T_1^T.$$

Thus, these limits are the same for the sequences (t_k) and (τ_k), which implies that the limits in (i) for $t \to 0+$ exist and are equal to the assigned values [which are therefore independent of the particular sequence (t_k), from the assumption].

Next, assertion (i) yields (iii), since

$$X(t) = T(t)\begin{pmatrix} I & 0 \\ 0 & D_2^{-1}(t) \end{pmatrix}T^T(t) \tag{17}$$

is regular whenever $D_2^{-1}(t)$ exists. Moreover, (15), (17) and the definition of $U(t)$ imply that

$$U(t)X^{-1}(t) = T(t)\begin{pmatrix} D_1(t) & 0 \\ 0 & I \end{pmatrix}T^T(t)T(t)\begin{pmatrix} I & 0 \\ 0 & D_2(t) \end{pmatrix}T^T(t) = Q(t),$$

which is assertion (iii).

Finally, the symmetry of $Q(t)$ [i.e. $U^T(t)X(t) \equiv X^T(t)U(t)$] and assertion (ii) show that $U^TX = X^TU$. Since

$$U = T\begin{pmatrix} D_1 & 0 \\ 0 & I \end{pmatrix}T^T, \qquad X = T\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}T^T,$$

we obtain that

$$\operatorname{rank}(U^T, X^T) = \operatorname{rank}\left\{\begin{pmatrix} D_1 & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right\} = m,$$

which completes the proof. ∎

4. EXAMPLES AND DISCUSSION OF ASSUMPTIONS

Here we discuss briefly the crucial assumptions in our results of Section 2 by stating some examples.

EXAMPLE 1. This example shows that the equality of the indices, i.e. the assumption (7) of Lemma 1, is needed for the assertion to be true in general.

Let $\alpha > 0$, and put

$$U = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$Q_1 = \frac{1}{2}\begin{pmatrix} -2\alpha & -1 \\ -1 & -1/\alpha \end{pmatrix}, \qquad Q_2 = \frac{1}{2}\begin{pmatrix} 2\alpha & -1 \\ -1 & 1/\alpha \end{pmatrix}.$$

Then (2) holds, and the matrices $K, \tilde{U}, \tilde{X}$ occurring in Lemma 1 are given by

$$K = \frac{1}{4}\begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}, \qquad \tilde{U} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{X} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

Now, all assumptions of Lemma 1 except (7)—in particular,

$$Q_2 - Q_1 = \begin{pmatrix} 2\alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} > 0$$

—are satisfied, while

$$S_1 - S_2 = \begin{pmatrix} -2\alpha & 2\alpha \\ 2\alpha & 2\alpha \end{pmatrix}$$

is indefinite, so that, of course, the assumption (7) cannot be true. Actually, we have that $0 = \operatorname{ind} \tilde{X}^T(\tilde{U} - Q_1\tilde{X}) \neq \operatorname{ind} \tilde{X}^T(\tilde{U} - Q_2\tilde{X}) = 1$.

EXAMPLE 2. This example concerns the assumption (4), i.e. the monotonicity of $Q(t)$, in Theorem 1. We shall see that (4) is needed for the limit result (5), and that, moreover, it cannot be replaced e.g. by (4′), which is sufficient for the assertion (6) according to Proposition 1. Put

$$U(t) = \frac{1}{2}\begin{pmatrix} t^3 - t & t^3 + t \\ 1 & -1 \end{pmatrix} \to U = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix},$$

and

$$X(t) = \begin{pmatrix} 1 & 1 \\ t^3 + t & t - t^3 \end{pmatrix} \to X = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{as} \quad t \to 0.$$

Then, for any $\varepsilon > 0$, the assumptions (1), (2), (3) of Theorem 1 are satisfied, but

$$Q(t) = U(t)X^{-1}(t) = \frac{1}{2t^3}\begin{pmatrix} t^6 + t^2 & -t \\ -t & 1 \end{pmatrix}$$

is not monotone [i.e., (4) is false]. Moreover, the limit result (5) does not hold, since

$$X^T Q(t) X = \frac{t^6 + t^2}{2t^3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

does not converge for $t \to 0$. Of course, by Proposition 1, the assertion (6) is true because $Q(t) > 0$ for $t > 0$ [i.e., (4′) holds with $\alpha = 0$].

EXAMPLE 3. As already mentioned in Remark 3, the one-sided boundedness, i.e. (4′), is needed for the assertion (6) of Proposition 1 and Theorem 1 to hold. Consider e.g.

$$U(t) \equiv U = I, \qquad X(t) = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \to X = 0 \quad \text{as } t \to 0.$$

Then (1), (2), and (3) hold, but

$$Q(t) = \begin{pmatrix} 1/t & 0 \\ 0 & -1/t \end{pmatrix}$$

is not one-sided bounded as $t \to 0+$, and (6) is not satisfied, since

$$c^T Q(t) c \equiv 0 \qquad \text{for} \quad c = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin \operatorname{Im} X = \{0\}.$$

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